Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 145 (2010) 2044–2054
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Notes

An additively separable representation in the Savage framework ☆

Brian Hill *

Department of Economics and Decision Sciences, HEC Paris, 1 rue de la Libération, 78351 Jouy-en-Josas, France

Received 8 October 2008; final version received 8 October 2009; accepted 3 March 2010

Available online 27 March 2010

Abstract

This paper proposes necessary and sufficient conditions for an additively separable representation of preferences in the Savage framework (where the objects of choice are acts: measurable functions from an infinite set of states to a potentially finite set of consequences). A preference relation over acts is represented by the integral over the subset of the product of the state space and the consequence space which corresponds to the act, where this integral is calculated with respect to an evaluation measure on this space. The result requires neither Savage's P3 (monotonicity) nor his P4 (weak comparative probability). Nevertheless, the representation it provides is as useful as Savage's for many economic applications.

JEL classification: D81

Keywords: Expected utility; Additive representation; State-dependent utility; Monotonicity

1. Introduction

According the standard theory of decision under uncertainty, first proposed and axiomatised by Savage [9], a rational agent's preferences over acts (measurable functions from an infinite set of states to a potentially finite set of consequences) are represented by a probability measure over states and a state-independent utility function on consequences. This representation

☆ The author would like to thank Itzhak Gilboa, Edi Karni, Philippe Mongin, Peter Wakker, the audience of RUD 2008, an anonymous referee and the associate editor for their useful comments.

* Fax: +33 1 39 67 71 09.
E-mail address: hill@hec.fr.
URL: http://www.hec.fr/hill.

0022-0531/$ – see front matter
doi:10.1016/j.jet.2010.03.011

assumes ordinal state-independence: that if one prefers one consequence to another, one will retain this preference no matter what event is realised. Moreover, it also assumes cardinal state-independence: that one's utility for a particular consequence is the same no matter what event is realised. These assumptions may been challenged.

On the one hand, it has long been recognised that cardinal state-independence does not hold in certain cases: for example, people's risk attitudes many depend on their state of health, and this may affect their choices regarding health and life insurance. On the other hand, although less commonly noted in the literature, ordinal state-independence also seems to fail in many cases, especially when the consequences are commodities. One might prefer a herbal tea at home to a dinner in an expensive restaurant next Saturday if tired, but the dinner to the tea if not. When in ill health, one might prefer living in a built-up area (to be close to the necessary services), whereas in good health, one might prefer the isolation of the countryside. These latter preferences inform one's decisions regarding investment in one's home, which, like health insurance decisions, require one to consider what one's state of health might be in the future and what preferences one would have in different possible states of health.

In such cases, the axioms corresponding to ordinal and cardinal state-independence – monotonicity (Savage's P3) and weak comparative probability (Savage's P4) respectively – are not satisfied. To account for such cases, a weakening of Savage's theory is required; namely, a set of axioms, including *neither* monotonicity *nor* weak comparative probability, which are necessary and sufficient for the following sort of *additively-separable* representation[1]: for all acts f, g,

$$f \preccurlyeq g \quad \text{iff} \quad \sum_{s \in S} U\big(s, f(s)\big) \leqslant \sum_{s \in S} U\big(s, g(s)\big) \tag{1}$$

where U is a real-valued function of states and consequences which we call the *evaluation*.[2] Under the standard theory, and in particular in the presence of the state-independence axioms, this function can be obtained as the product of the probability and the state-independent utility. However, in cases of possible state-dependence, probability and state-independent utility functions do not always exist, and even when they do, they lack economic and empirical meaning because they are not unique in the appropriate sense (see [5,6]). By contrast, the evaluation U exists even in cases of state-dependence and it does have the appropriate uniqueness properties; in cases of state-dependence, it is the more solid economically meaningful concept.

Moreover, it can fill many of the roles traditionally played by probability and utility in economic applications. Consider, for example, the decision maker's choice today about whether to book a table at the restaurant next Saturday or not. To determine what he will choose, only his evaluation U is needed: a factorisation into a probability measure and a utility function adds no useful information. Furthermore, the representation of preferences by an evaluation is entirely sufficient for most standard exercises of comparative statics. For example, the evaluation is all that is needed to determine the effect of changes in the restaurant's prices on the decision maker's choice.

To date, the only axiomatisation of the form (1) in the Savage framework [11] drops weak comparative probability but retains monotonicity, and hence does not apply in the cases mentioned above. In this paper, we provide an axiomatisation in the Savage framework which relies

[1] In this formula, we assume that there are finitely many states. This assumption is made entirely for expositional purposes: it is not true in the Savage framework, nor in the result proposed here. See Sections 2 and 3.

[2] Thanks to Mark Machina and Edi Karni for suggesting this terminology.

neither on monotonicity nor on weak comparative probability. The theorem is stated in Section 2 and is discussed, along with related literature, in Section 3. The proof is to be found in Appendix A.

2. Axioms and theorem

Let S be a set of states, with a σ-algebra of events $\mathcal{F}_S$ which contains $\{s\}$ for each $s \in S$, and let C be a set of consequences, with a σ-algebra $\mathcal{F}_C$. C can be finite or infinite: if it is finite or countable, $\mathcal{F}_C$ will just be the set of subsets of C; if it is uncountable, this need not be the case. Let $\mathcal{A}$ be the set of measurable functions from S to C – they are called the *acts* – and let $\preccurlyeq$ be a binary relation on $\mathcal{A}$; $\prec$, $\sim$, $\succ$ and $\succcurlyeq$ are defined from $\preccurlyeq$ in the usual way.

Let $\mathcal{A}_p$ be the set of partial measurable functions[3] from S to C; that is, measurable functions from A to C, where $A \subseteq S$. Elements of $\mathcal{A}_p$ are called *partial acts*, and the set A on which a partial act is defined is called its *domain*. Note that, since A can be S, every act belongs to $\mathcal{A}_p$. For any act f and event A, f_A will denote the partial act with domain A which agrees with f on this domain. To each partial act f_A, there corresponds a subset of $S \times C$: namely its graph $\{(s, f_A(s)) \mid s \in A\}$. This subset shall also be called f_A. $\mathcal{F}_{SC}$ is defined to be the σ-algebra on $S \times C$ generated by $\mathcal{A}_p$. Note that, when C is finite or countable, $\mathcal{F}_{SC}$ coincides with the product of $\mathcal{F}_S$ and $\mathcal{F}_C$; however, if C is uncountable, this need not be the case.[4]

For A and B disjoint, $f_A g_B$ is the partial function taking the values of f on A and the values of g on B. Given an event A, A^c is the set $S \setminus A$; since it is measurable, it is also an event. As is standard, $\preccurlyeq_A$ will denote the order $\preccurlyeq$ on acts, given the event A. The traditional notion of null event shall be employed: an event A is *null* iff, for any pair of acts $f, g \in \mathcal{A}$, $f \sim_A g$.

We assume two of the basic axioms of Savagean decision theory.

Axiom A1 *(Weak order).* $\preccurlyeq$ is a weak order: (a) for all f, g in $\mathcal{A}$, $f \preccurlyeq g$ or $g \preccurlyeq f$; and (b) for all f, g and h in $\mathcal{A}$, if $f \preccurlyeq g$ and $g \preccurlyeq h$, then $f \preccurlyeq h$.

Axiom A2 *(Sure-thing principle).* For any acts f, g, h, h' in $\mathcal{A}$ and any non-null event A, $f_A h_{A^c} \preccurlyeq g_A h_{A^c}$ iff $f_A h'_{A^c} \preccurlyeq g_A h'_{A^c}$.

Following [1,3], we factorise Savage's P6 axiom into two elements: the standard Archimedean axiom (see for example [8, pp. 204–205]) and solvability (which differs slightly from the solvability axioms proposed in the articles mentioned, in so far as they suppose monotonicity, and this is not assumed here).

Axiom A3 *(Archimedean axiom).* For each pair of acts f and g, if $A_1, \ldots, A_i, \ldots$ is a sequence of disjoint non-null events such that $f_{A_i} g_{A_i^c} \sim f_{A_j} g_{A_j^c} \succ g$ for all i, j, then it is a finite sequence.

Axiom A4 *(Solvability).* For acts f, g, h in $\mathcal{A}$ with $f \prec g \prec h$, there exists an event $A \subseteq S$ such that $f_A h_{A^c} \sim g$.

---

[3] Henceforth all functions, partial functions and sets of states shall be assumed to be measurable.

[4] Suppose that $S = C = \Re$ with $\mathcal{F}_S$ and $\mathcal{F}_C$ the Borel sets: on the one hand, the graphs of strictly monotonic measurable functions are $\mathcal{F}_{SC}$-measurable but not $\mathcal{F}_S \times \mathcal{F}_C$-measurable; on the other hand, subsets $\{(s, c) \mid s \in (s_1, s_2), c \in (c_1, c_2)\}$ for $s_1 < s_2$ and $c_1 < c_2$ are $\mathcal{F}_S \times \mathcal{F}_C$-measurable but not $\mathcal{F}_{SC}$-measurable.

A final technical condition is required, which was first proposed in [4].

Axiom A5 *(Local stability).* For any f, g in $\mathcal{A}$, there exist events A_1, A_2 and A_3, such that the non-empty A_i form a partition of S and such that: for any non-null event $B \subseteq A_1$, $f \prec_B g$; for any non-null event $B \subseteq A_2$, $f \sim_B g$; and for any non-null event $B \subseteq A_3$, $f \succ_B g$.

This axiom is required because the set of states is infinite; it is automatically satisfied on finite sets of states.

Theorem 1. *For a given S, C, $\preccurlyeq$, suppose that* A4 *and* A5 *are satisfied. Then the following are equivalent*:

(i) $\preccurlyeq$ *satisfies* A1–A3;
(ii) *there exists a measure U on $(S \times C, \mathcal{F}_{SC})$ which takes finite values on $\mathcal{A}$ and is such that, for every $f, g \in \mathcal{A}$,*

$$f \preccurlyeq g \quad \text{iff} \quad \int_f dU \leqslant \int_g dU \tag{2}$$

Furthermore, let U' be any other measure satisfying (2). *Then there exists $a > 0$ and a measure b on $(S, \mathcal{F}_S)$ such that $U' = aU + b$.*

U is the *evaluation* and can be thought of as the analogue of the function U in (1). For further discussion of this representation, see Section 3.

3. Discussion

Axiomisations for representation (1) have been obtained long ago in frameworks involving a finite number of states and an infinite set of consequences endowed with a rich structure: for example, where the set of consequences is the set of lotteries over a set of outcomes (the so-called Anscombe and Aumann framework) or a connected topological space. However little work has been done on how such a representation can be got in the Savage framework, which assumes a possibly finite set of consequences and an infinite set of states. The main paper in the literature dealing with this problem is Wakker and Zank [11]. We shall bring out the pertinent aspects of Theorem 1 via the comparison with their proposal.[5]

The most important difference between the two papers lies in the use of monotonicity: Wakker and Zank need this axiom, whereas Theorem 1 does not assume it. As noted in Section 1, this axiom is violated in several interesting cases, and any full reply to the problem of state-dependent utilities should be able to deal with such cases. The theorem proposed here does apply in such cases, whereas Wakker and Zank's does not.

Another important difference is in the assumptions on the set of consequences. Wakker and Zank require that the set of consequences is a connected topological space, whereas no structure is assumed on the set of consequences here. In particular, Theorem 1 applies to finite, as

[5] The main points made below also hold for [2], which proposes results similar to those in [11], though with a different motivation.

well as infinite, sets of consequences. Not only is this closer to Savage's original theory, where the structural burden is borne by the state space alone, but there are many cases, in particular those involving indivisible goods, where the assumptions on the set of consequences made by Wakker and Zank may be difficult to justify, but where the theorem proposed here continues to apply.

One final difference between the two papers lies in the presentation of the result. It is a non-trivial conceptual problem to find an analogue of (1) when the set of states is infinite (see [11] for an excellent discussion). Wakker and Zank consider two solutions of this problem, both of which are different from, though of course equivalent to, the solution proposed here. To illustrate the relationship, consider Wakker and Zank's second solution [11, Theorem 12], which represents preferences by a functional of the form $\int_S u(s, f(s))\, d\mu(s)$, where μ is a non-unique measure on the state space, and u is a function on state-consequence pairs which is not unique up to positive affine transformations. Firstly, the representation proposed here, which involves a single measure U on the Cartesian product of the state space and the consequence space retains a closer analogy with (1), the only difference being the replacement of the sum by an integral and the function by a measure. Moreover, given the uniqueness of the evaluation U and the lack of uniqueness of the functions featuring in Wakker and Zank's representation, the former reflects more accurately the economic and empirical content of the representation.

The first two differences mentioned above render Theorem 1 more useful in the construction of new theories of state-dependent utilities. The goal of much of the literature on state-dependence is to find necessary and sufficient conditions for the existence of a unique probability function and a suitably unique state-dependent utility function which represents the decision maker's preferences. A popular technique is to assume axioms necessary and sufficient for a representation of the form (1), and then search for new supplementary conditions which guarantee a unique decomposition of the evaluation U (for example [6,7]). To date, most of this research has had to adopt a framework with a finite state space and a rich consequence space, because, as noted above, these are the only frameworks in which (1) has been axiomatised without relying on monotonicity. Using Theorem 1, one can now expect to be able to extend these theories to the Savage framework.

Finally, the result may have useful applications to time preferences. If one interprets S as a set of time points (say, an interval on the real line) and C as a set of consumption bundles, then the objects of choice (acts) are consumption streams of the sort introduced by Strotz [10]. Theorem 1 applies, yielding a representation of the form (2). This representation bears the same relationship to the functional form proposed by Strotz as representation (1) bears to the representation proposed by state-dependent utility theorists such as [7]: the single measure on consumption bundle-time point pairs is decomposed into a discount factor and a time-dependent "instantaneous" utility function [10, p. 167]. And just like for the case of state-dependent utility, all that is required to obtain Strotz's representation are supplementary axioms which guarantee the decomposition. To the knowledge of the author, this is the first axiomatisation in the continuous-time setting of a functional form close to that proposed by Strotz: in particular, contrary to much of the recent literature on time-discounting, it does not suppose that "instantaneous utilities" are time-independent (an assumption which corresponds to state-independence).

Appendix A

Preliminary notions. The proof relies heavily on Theorem 4 in [8, Ch 2]. It is worth reproducing the essential definition and the statement of the theorem.

Definition 1. (See [8, p. 44].) Let A be a non-empty set, B and $\succsim$ non-trivial binary relations on A and $\circ$ a binary operation from B into A. The quadruple $\langle A, \succsim, B, \circ \rangle$ is an *Archimedean, regular, positive, ordered local semigroup* iff, for all $a, b, c, d \in A$, the following eight axioms are satisfied:

1. $\langle A, \succsim \rangle$ is a total order: that is, an anti-symmetric weak order (a weak order such that, if $a \succcurlyeq b$ and $b \succcurlyeq a$, then $a = b$).
2. If $(a, b) \in B$, $a \succsim c$, and $b \succsim d$, then $(c, d) \in B$.
3. If $(c, a) \in B$ and $a \succsim b$, then $c \circ a \succsim c \circ b$.
4. If $(a, c) \in B$ and $a \succsim b$, then $a \circ c \succsim b \circ c$.
5. $(a, b) \in B$ and $(a \circ b, c) \in B$ iff $(b, c) \in B$ and $(a, b \circ c) \in B$; and when both conditions hold $(a \circ b) \circ c = a \circ (b \circ c)$.
6. If $(a, b) \in B$, then $a \circ b \succ a$.
7. If $a \succ b$, then there exists $c \in A$ such that $(b, c) \in B$ and $a \succsim b \circ c$.
8. $\{n \mid n \in N_a \text{ and } b \succ na\}$ is a finite set.

Where N_a, a subset of the positive integers, and na, an element of A for each $n \in N_a$, are defined inductively as follows:

(i) $1 \in N_a$ and $1a = a$;
(ii) if $n - 1 \in N_a$ and $((n-1)a, a) \in B$, then $n \in N_a$ and na is defined to be $((n-1)a) \circ a$;
(iii) if $n - 1 \in N_a$ and $((n-1)a, a) \notin B$, then for all $m \geqslant n$, $m \notin N_a$.

The importance of this definition is expressed by the following theorem (Theorem 4 in [8]).

Theorem 2. *Let $\langle A, \succsim, B, \circ \rangle$ be an Archimedean, regular, positive, ordered local semigroup. Then there is a function ϕ from A to $\Re_+$ such that for all $a, b \in A$,*

(i) *$a \succsim b$ iff $\phi(a) \geqslant \phi(b)$;*
(ii) *if $(a, b) \in B$, then $\phi(a \circ b) = \phi(a) + \phi(b)$.*

Moreover, if ϕ and ϕ' are two functions from A to $\Re_+$ satisfying conditions (i) *and* (ii)*, then there exists $\alpha > 0$ such that, for any non-maximal $a \in A$, $\phi'(a) = \alpha\phi(a)$.*

Proof of Theorem 1. The proof employs several propositions and lemmas which are proved separately below. The (ii) to (i) direction is straightforward, so we will restrict attention to the (i) to (ii) direction. Henceforth we assume that A1–A5 hold.

Pick any act $e \in \mathcal{A}$, which shall remain fixed throughout the proof. Let $\mathcal{A}^+ = \{f_A \in \mathcal{A}_p \mid A$ non-null event and for any non-null event $A' \subseteq A$, $f_{A'}e_{A'^c} \succ e\}$ and $\mathcal{A}^- = \{f_A \in \mathcal{A}_p \mid A$ non-null event and for any non-null event $A' \subseteq A$, $e \succ f_{A'}e_{A'^c}\}$. Consider the relation $\sim$ on $\mathcal{A}^+$ defined by $f_A \sim g_B$ if $f_A e_{A^c} \sim g_B e_{B^c}$; by A1, this is an equivalence relation. Let $\mathcal{A}^+_\sim$ be the set of equivalence classes of $\mathcal{A}^+$ under $\sim$. $\mathcal{A}^-_\sim$ is defined similarly. $[f_A]$ shall denote the equivalence class (element of $\mathcal{A}^+_\sim$) containing f_A.[6] Define the order $\precsim^+_\sim$ on $\mathcal{A}^+_\sim$ as follows: $[f_A] \precsim^+_\sim [g_B]$ iff,

[6] So if $f_A \sim g_B$ then $[f_A] = [g_B]$. Often, below, we shall need to chose an arbitrary element from an equivalence class $[f_A]$; for notational convenience we shall often call the element f_A.

for any $f_A \in [f_A]$ and $g_B \in [g_B]$, $f_A e_{A^c} \preccurlyeq g_B e_{B^c}$. Define $\mathcal{B}^+$, a set of sequences of elements of $\mathcal{A}^+_\sim$, as follows: for I countable or finite of size greater than or equal to 2, $([f^i_{A_i}])_{i\in I} \in \mathcal{B}^+$ iff there exists, for each $i \in I$, $f^i_{A_i} \in [f^i_{A_i}]$ such that the A_i are pairwise disjoint. Then define $\circ^+$, a function from $\mathcal{B}^+$ into $\mathcal{A}^+_\sim$, as follows: $\circ^+([f^i_{A_i}])_{i\in I} = [f^1_{A_1} \cdots f^i_{A_i} \cdots e_{(\cup A_i)^c}]$ where for each $i \in I$, $f^i_{A_i} \in [f^i_{A_i}]$ and the A_i are pairwise disjoint. Henceforth we will write $\bigcirc^+_{i\in I}[f^i_{A_i}]$ instead of $\circ^+(([f^i_{A_i}])_{i\in I})$. Note that the restriction of $\mathcal{B}^+$ to sequences of length 2 is a relation on $\mathcal{A}^+_\sim$, and the restriction of $\circ^+$ to such sequences is a binary operation on $\mathcal{A}^+_\sim$; in such cases we will write $[f_A] \circ^+ [g_B]$ rather than $\bigcirc^+([f_A], [g_B])$. $\mathcal{B}^-$ is defined similarly.

Let us first consider the trivial cases. If $\mathcal{A}^+_\sim$ and $\mathcal{A}^-_\sim$ are both empty, then define U to be the measure on $(S \times C, \mathcal{F}_{SC})$ which is 0 everywhere. If one or both of $\mathcal{A}^+_\sim$ and $\mathcal{A}^-_\sim$ are non-empty, but $\mathcal{B}^+$ and $\mathcal{B}^-$ are empty, then $\mathcal{A}^+_\sim$ (resp. $\mathcal{A}^-_\sim$) has one element if it is non-empty (Lemma 2). In such cases, define the measure U on $(S \times C, \mathcal{F}_{SC})$, by $U(f_A) = 0$ if $f_A e_{A^c} \sim e$, $U(f_A) = 1$ if $f_A e_{A^c} \succ e$ and $U(f_A) = -1$ if $f_A e_{A^c} \prec e$ for any $f_A \in \mathcal{A}_p$. It is straightforward to show in both these cases that U represents $\succcurlyeq$ according to (2).

For the rest of the proof it will be assumed that one or both of $\mathcal{B}^+$ and $\mathcal{B}^-$ are non-empty. The proof for this case rests on the following proposition.

Proposition 1. *$\langle \mathcal{A}^+_\sim, \succcurlyeq^+_\sim, \mathcal{B}^+, \circ^+\rangle$ and $\langle \mathcal{A}^-_\sim, \preccurlyeq^-_\sim, \mathcal{B}^-, \circ^-\rangle$ are Archimedean regular positive ordered local semigroups* (*Definition* 1).

It follows from Theorem 2 that there is a function $U^+ : \mathcal{A}^+_\sim \to \Re^+$ which respects the order $\succcurlyeq^+_\sim$ (for $[f_A], [g_B] \in \mathcal{B}^+$, $U^+([f_A]) \geqslant U^+([g_B])$ iff $[f_A] \succ^+_\sim [g_B]$) and represents a finite number of applications of the operation $\circ^+$ by addition (for $[f_A], [g_B] \in \mathcal{B}^+$, $U^+([f_A] \circ^+ [g_B]) = U^+([f_A]) + U^+([g_B])$); moreover, this function is unique up to a positive multiplicative factor. Similarly there is a function $U^- : \mathcal{A}^-_\sim \to \Re^+$ inversing order (for $[f_A], [g_B] \in \mathcal{B}^-_\sim$, $U^-([f_A]) \geqslant U^-([g_B])$) iff $[f_A] \preccurlyeq^-_\sim [g_B]$ and representing a finite number of applications of $\circ^-$ by addition, which is unique up to a positive multiplicative factor. (Equivalently, $-U^-$ respects the order $\succcurlyeq^-_\sim$ and represents $\circ^-$ by addition.) These naturally induce real-valued functions on $\mathcal{A}^+$ and $\mathcal{A}^-$ sharing the same properties; to ease notation, these functions will also be called U^+ and U^- (so we have $U^+(f_A) = U^+([f_A])$).

As is clear from the proof of Theorem 2 [8, §2.2.2], there are two cases to be considered. In the first case, there is a smallest element $[h_C]$ of $\mathcal{A}^+_\sim$ (for all $[f_A] \in \mathcal{A}^+_\sim$, $[h_C] \preccurlyeq^+_\sim [f_A]$), and every element of $\mathcal{A}^+_\sim$ is of the form $n[h_C]$ for some positive integer n [8, §2.2.2], so $\mathcal{A}^+_\sim$ is finite and there are no infinite sequences in $\mathcal{B}^+$. In the second case, there is no smallest element of $\mathcal{A}^+_\sim$, so $\mathcal{B}^+$ may contain infinite sequences. Proposition 1 only guarantees that U^+ is finitely additive – i.e. it represents finite applications of $\circ^+$ by addition. Although this suffices for the first case, it still remains to be shown in the second case that U^+ is countably additive – i.e. that it represents the application of $\circ^+$ on infinite sequences by addition. Similarly for U^-. This is indeed the case.

Proposition 2. *Suppose that $\mathcal{A}^+_\sim$ (resp. $\mathcal{A}^-_\sim$) does not have a smallest element. Then U^+ (resp. U^-) is countably additive; that is, for $([g^i_{B_i}])_{i\in\mathbb{N}} \in \mathcal{B}^+$, $U^+(\bigcirc^+_{i\in\mathbb{N}}([g^i_{B_i}])) = \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$.*

It remains to "calibrate" the functions U^+ and U^-; that is, to ensure that the positive and negative utilities add correctly. This only needs to be done if both $\mathcal{A}^+_\sim$ and $\mathcal{A}^-_\sim$ are non-empty; henceforth we suppose that this is indeed the case.

Definition 2. $[f_A] \in \mathcal{A}_\sim^+$ and $[g_B] \in \mathcal{A}_\sim^-$ *cancel* if there exist $f_A \in [f_A]$, $g_B \in [g_B]$, such that A and B are disjoint and $f_A g_B e_{(A\cup B)^c} \sim e$.

Let $\mathcal{I}^+ = \{[f_A] \in \mathcal{A}_\sim^+ \mid \text{there exists } [g_B] \in \mathcal{A}_\sim^-,\ [f_A] \text{ and } [g_B] \text{ cancel}\}$ and let $\mathcal{I}^- = \{[g_B] \in \mathcal{A}_\sim^- \mid \text{there exists } [f_A] \in \mathcal{A}_\sim^+,\ [f_A] \text{ and } [g_B] \text{ cancel}\}$. These are non-empty, given that $\mathcal{A}_\sim^+$ and $\mathcal{A}_\sim^-$ are (Lemma 3). Moreover, there is a natural mapping $\sigma : \mathcal{I}^+ \to \mathcal{I}^-$, taking $[f_A]$ to the $[g_B]$ such that $[f_A]$ and $[g_B]$ cancel. It is straightforward to show that this mapping is well defined, and that it is one-to-one and onto.

We shall refer to the restriction of $\mathcal{B}^+$ to $\mathcal{I}^+$ by $\mathcal{B}_\mathcal{I}{}^+$ and the restrictions of $\succcurlyeq_\sim^+$ and $\circ^+$ by $\succcurlyeq_\sim^+$ and $\circ^+$ respectively; similarly for $\mathcal{I}^-$. Note that $\mathcal{B}_\mathcal{I}{}^+$ is non-empty if and only if $\mathcal{B}_\mathcal{I}{}^-$ is, so there are two cases.

Case 1. $\mathcal{B}_\mathcal{I}{}^+$ and $\mathcal{B}_\mathcal{I}{}^-$ are empty. In this case, $\mathcal{I}^+$ and $\mathcal{I}^-$ contain one element each (the proof of this is a simple extension of the proof of Lemma 2); let the elements be $\widehat{[f_A]}$ and $\sigma(\widehat{[f_A]})$ respectively. So for $[g_B] \in \mathcal{I}^-$, $U^-([g_B]) = \alpha U^+(\sigma^{-1}([g_B]))$, where $\alpha = \frac{U^-(\sigma(\widehat{[f_A]}))}{U^+(\widehat{[f_A]})}$.

Case 2. $\mathcal{B}_\mathcal{I}{}^+$ and $\mathcal{B}_\mathcal{I}{}^-$ are non-empty. By inspection of the proof of Proposition 1, it is easily seen that $\langle \mathcal{I}^+, \succcurlyeq_\sim^+, \mathcal{B}_\mathcal{I}{}^+, \circ^+\rangle$ and $\langle \mathcal{I}^-, \preccurlyeq_\sim^-, \mathcal{B}_\mathcal{I}{}^-, \circ^-\rangle$ are Archimedean regular positive ordered local subgroups, so Theorem 2 applies. U^- represents the restriction of $\preccurlyeq_\sim^-$ to $\mathcal{I}^-$; however, $U^+.\sigma^{-1}$ does as well.

Proposition 3. *For* $[g_B], [g'_{B'}] \in \mathcal{I}^-$, $U^+(\sigma^{-1}([g_B])) \geqslant U^+(\sigma^{-1}([g'_{B'}]))$ *iff* $[g_B] \preccurlyeq_\sim^- [g'_{B'}]$.

Since, by Theorem 2, any two representations of the restriction of $\preccurlyeq_\sim^-$ to $\mathcal{I}^-$ are related by a positive multiplicative transformation, there exists an $\alpha > 0$ such that, for all $[g_B] \in \mathcal{I}^-$, $U^-([g_B]) = \alpha U^+(\sigma^{-1}([g_B]))$.

Define $U : \mathcal{A}_p \to \Re$ as follows. Firstly define it on $\mathcal{A}^+ \cup \mathcal{A}^- \cup \{f_A \in \mathcal{A}_p \mid \text{for any event } A' \subseteq A,\ f_{A'}e_{A'^c} \sim e\}$:

$$U(f_A) = \begin{cases} U^+([f_A]) & \text{if } f_{A'}e_{A'^c} \succ e \text{ for all non-null events } A' \subseteq A \\ -\frac{1}{\alpha}U^-([f_A]) & \text{if } f_{A'}e_{A'^c} \prec e \text{ for all non-null events } A' \subseteq A \\ 0 & \text{if } f_{A'}e_{A'^c} \sim e \text{ for all events } A' \subseteq A \end{cases} \tag{3}$$

For any $f_A \in \mathcal{A}_p$, A5 applied to $f_A e_{A^c}$ and e ensures that there is a partition of A into three events A_1, A_2, A_3 such that, on each event, one of the three conditions in (3) holds. So U can be defined for any $f_A \in \mathcal{A}_p$ by $U(f_A) = U(f_{A_1}) + U(f_{A_2}) + U(f_{A_3})$.

Given that $\mathcal{F}_{SC}$ is the σ-algebra generated by $\mathcal{A}_p$, this suffices to define U on $\mathcal{F}_{SC}$. By construction, U is a countably additive measure on $(S \times C, \mathcal{F}_{SC})$, which is finite-valued on $\mathcal{A}$ and represents $\succcurlyeq$ according to (2).

Uniqueness. Assume that $\succcurlyeq$ is non-trivial and let U' be any other measure on $(S \times C, \mathcal{F}_{SC})$ taking finite values on $\mathcal{A}$ and representing $\succcurlyeq$. Define $b : \mathcal{F}_S \to \Re$ by $b(A) = U'(e_A)$, where e is the act used in the construction of U (e is such that, for any event A, $U(e_A) = 0$). By the additivity properties of U', b is a finite-valued measure on $(S, \mathcal{F}_S)$. Consider the measure $U' - b.p_1$ on $(S \times C, \mathcal{F}_{SC})$, where p_1 is the projection onto S: $p_1(f_A) = A$. For any $f_A \in \mathcal{A}^+$ and for all non-null $A' \subseteq A$, $f_{A'} \succ e_{A'}$, so $U'(f_{A'}) > U'(e_{A'})$ and hence $(U' - b.p_1)(f_{A'}) > 0$. $U' - b.p_1$ is thus strictly positive on $\mathcal{A}^+$. Furthermore, $U' - b.p_1$ is well defined on the set of equivalence classes $\mathcal{A}^+_\sim$; we now show that $U' - b.p_1$ represents $\succsim^+_\sim$. Let $[f_A], [g_B] \in \mathcal{A}^+_\sim$ and suppose that $[f_A] \succsim^+_\sim [g_B]$. By Lemma 1, there exists non-null $A' \subseteq A$ such that $f_{A'} \in [g_B]$. But $(U'-b.p_1)([f_A]) - (U'-b.p_1)([g_B]) = (U'-b.p_1)(f_A) + (U'-b.p_1)(e_{A^c}) - (U'-b.p_1)(f_{A'}) - (U'-b \circ p_1)(e_{A'^c}) = U'(f_{A\setminus A'}) - U'(e_{A\setminus A'}) \geqslant 0$ since $f_{A\setminus A'}e_{(A\setminus A')^c} \succsim e$. Similarly, if $(U'-b.p_1)([f_A]) - (U'-b.p_1)([g_B]) \geqslant 0$ then $U'(f_A e_{A^c}) \geqslant U'(g_B e_{B^c})$ and hence $[f_A] \succsim^+_\sim [g_B]$, since U' represents $\succsim$. So $U' - b.p_1$ represents $\succsim^+_\sim$; a similar argument holds for $\precsim^-_\sim$ and $\mathcal{A}^-_\sim$. By the uniqueness clause in Theorem 2, $(U' - b \circ p_1) = a.U$ for some $a > 0$.

This completes the proof of Theorem 1.

Proofs of auxiliary propositions and lemmas. The following lemmas will be used below or have been mentioned above.

Lemma 1. *For $f_A, g_B \in \mathcal{A}^+$, if $[f_A] \succsim^+_\sim [g_B]$, then there exists a non null event $A' \subseteq A$, such that $f_{A'} \in [g_B]$.*

Proof. If $f_A e_{A^c} \sim g_B e_{B^c}$, let $A' = A$. If not, $f_A e_{A^c} \succ g_B e_{B^c} \succ e$, so, by A4, there is an event $A'' \subseteq A$ such that $f_{A\setminus A''} e_{A^c \cup A''} \sim g_B e_{B^c}$. Setting $A' = A \setminus A''$ yields the required result. If A' were null, then $g_B e_{B^c} \sim f_{A'} e_{A'^c} \sim e$, contradicting $g_B \in \mathcal{A}_+$. □

Lemma 2. *If $\mathcal{A}^+_\sim$ (resp. $\mathcal{A}^-_\sim$) is non-empty and $\mathcal{B}^+$ (resp. $\mathcal{B}^+$) is empty, then $\mathcal{A}^+_\sim$ (resp. $\mathcal{A}^-_\sim$) has just one element.*

Proof. Suppose not, and take $f_A, g_B \in \mathcal{A}^+$ with $[f_A] \succ^+_\sim [g_B]$. By Lemma 1, there is a non null event $A' \subseteq A$ such that $f_{A'} \in [g_B]$, but by A2 and the fact that $[f_A] \neq [g_B]$, $A \setminus A'$ must also be non null. $([f_{A'}], [f_{A\setminus A'}]) \in \mathcal{B}^+$, contradicting the assumption that it was empty. □

Lemma 3. *If $\mathcal{A}^+_\sim$ and $\mathcal{A}^-_\sim$ are non-empty, then $\mathcal{I}^+$ and $\mathcal{I}^-$ are non-empty.*

Proof. Consider arbitrary $f_A \in \mathcal{A}^+$ and $g_B \in \mathcal{A}^-$. If A and B have non-null intersection, then Axiom A4 applied to $f_A e_{A^c} \succ e \succ g_B e_{B^c}$ yields $f_{A'} g_{B'} e_{(A' \cup B')^c} \sim e$, where $A' \subseteq A$, $B' \subseteq B$ are non-null events; so $[f_{A'}]$ and $[g_{B'}]$ cancel. If A and B have null intersection then they can be assumed to be disjoint (remove the intersection from one). Consider $f_A g_B e_{(A\cup B)^c}$: if $f_A g_B e_{(A\cup B)^c} \sim e$ then $[f_A]$ and $[g_B]$ cancel; if $f_A g_B e_{(A\cup B)^c} \succ e$, then applying A4 to $f_A g_B e_{(A\cup B)^c} \succ e \succ g_B e_{B^c}$ gives $f_{A'} g_{B'} e_{(A'\cup B')^c} \sim e$ where $A' \subseteq A$ and $B' \subseteq B$ so $[f_{A'}]$ and $[g_{B'}]$ cancel; similarly for $f_A g_B e_{(A\cup B)^c} \prec e$. □

Proof of Proposition 1. Since the cases are similar, we shall only treat the case of $\langle \mathcal{A}^+_\sim, \succsim^+_\sim, \mathcal{B}^+, \circ^+ \rangle$ here. We shall show that the clauses of Definition 1 are satisfied:

1. *$\precsim^+_\sim$ is a total order.* The order $\precsim^+_\sim$ on $\mathcal{A}^+_\sim$ inherits the properties of connectedness and transitivity from the order $\precsim$ on $\mathcal{A}$: so axiom A1 guarantees that $\precsim^+_\sim$ is a weak order. Furthermore, since $\mathcal{A}^+_\sim$ is obtained by quotienting by $\sim$, $\precsim^+_\sim$ is anti-symmetric. It is thus a total order.

2. *If $([f_A], [g_B]) \in \mathcal{B}^+$, $[f_A] \succsim^+_\sim [f'_{A'}]$ and $[g_B] \succsim^+_\sim [g'_{B'}]$, then $([f'_{A'}], [g'_{B'}]) \in \mathcal{B}^+$.* Since $([f_A], [g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. Using Lemma 1, choose events $A'' \subseteq A$ and $B'' \subseteq B$ such that $f_{A''} \in [f'_{A'}]$ and $g_{B''} \in [g'_{B'}]$. Since A and B are disjoint, so are A'' and B'', and hence $([f'_{A'}], [g'_{B'}]) \in \mathcal{B}^+$.

3 and 4. Note that, since $\circ^+$ is commutative, Clause 3 is satisfied if and only if Clause 4 is. So it suffices to establish that: *If $([f_A], [g_B]) \in \mathcal{B}^+$ and $[f_A] \succsim^+_\sim [h_C]$, then $[f_A] \circ^+ [g_B] \succsim^+_\sim$*

$[h_C] \circ^+ [g_B]$. Since $([f_A],[g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. Using Lemma 1, choose an event $A' \subseteq A$ such that $f_{A'} \in [h_C]$. By the definition of $\mathcal{A}^+_{\precsim}$, $f_{A'}e_{A'^c} \preccurlyeq f_A e_{A^c}$; by Axiom A2, it follows that $f_{A'}g_B e_{(A' \cup B)^c} \preccurlyeq f_A g_B e_{(A \cup B)^c}$. Hence $[h_C] \circ^+ [g_B] \precsim^+_{\sim} [f_A] \circ^+ [g_B]$.

5. $([f_A],[g_B]) \in \mathcal{B}^+$ *and* $([f_A] \circ^+ [g_B],[h_C]) \in \mathcal{B}^+$ *iff* $([g_B],[h_C]) \in \mathcal{B}^+$ *and* $([f_A],[g_B] \circ^+ [h_C]) \in \mathcal{B}^+$*; and when both conditions hold* $([f_A] \circ^+ [g_B]) \circ^+ [h_C] = [f_A] \circ^+ ([g_B] \circ^+ [h_C])$. If $([f_A],[g_B]) \in \mathcal{B}^+$ and $([f_A] \circ^+ [g_B],[h_C]) \in \mathcal{B}^+$, then there are f_A, g_B and h_C, members of $[f_A]$, $[g_B]$ and $[h_C]$ respectively, such that A, B and C are disjoint. It follows that $([g_B],[h_C]) \in \mathcal{B}^+$ and $([f_A],[g_B] \circ^+ [h_C]) \in \mathcal{B}^+$; furthermore $([f_A] \circ^+ [g_B]) \circ^+ [h_C] = [f_A g_B h_C] = [f_A] \circ^+ ([g_B] \circ^+ [h_C])$. The same argument works in the other direction.

6. *If* $([f_A],[g_B]) \in \mathcal{B}^+$*, then* $[f_A] \circ^+ [g_B] \succ^+_{\sim} [f_A]$. Since $([f_A],[g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. $f_A g_B e_{(A \cup B)^c}$ and $f_A e_{A^c}$ differ solely on B; by A2, it is their comparison on this set that decides the preference ordering between them. Furthermore, by definition of $\mathcal{A}^+$, $g_B e_{B^c} \succ e$; hence the required result.

7. *If* $[f_A] \succ^+_{\sim} [g_B]$*, then there exists a* $[h_C] \in \mathcal{A}^+_{\sim}$ *with* $([g_B],[h_C]) \in \mathcal{B}^+$ *and* $[f_A] \succsim^+_{\sim} [g_B] \circ^+ [h_C]$. By Lemma 1, there is an $A' \subseteq A$ such that $f_{A'} \in [g_B]$. Since $f_A e_{A^c} \succ f_{A'} e_{A'^c}$, $A \setminus A'$ is a non-null event; take C to be an arbitrary non-null subevent of $A \setminus A'$. Since $f_A \in \mathcal{A}^+$ and C is non-null, $f_C \in \mathcal{A}^+$; since $A' \cup C \subseteq A$, $f_{A' \cup C} e_{(A' \cup C)^c} \preccurlyeq f_A e_{A^c}$. Taking $[h_C] = [f_C]$ gives the result.

8. *For all* $[f_A],[g_B] \in \mathcal{A}^+_{\sim}$, $\{n \mid n \in N$ *and* $[g_B] \succ n[f_A]\}$ *is finite, where* $N_{[f_A]}$ *and* $n[f_A]$ *as in Definition* 1. Suppose not. Then there exists an infinite sequence $f^i_{A_i} \in \mathcal{A}^+$ such that for all i, j, $f^i_{A_i} e_{A_i^c} \sim f^j_{A_j} e_{A_j^c} \succ e$ and the A_i are disjoint. The acts $f = f^1_{A_1} \dots f^i_{A_i} \dots e_{(\bigcup A_i)^c}$ and e, and the infinite sequence of disjoint non-null events A_i violate A3.

This concludes the proof of Proposition 1. □

Proof of Proposition 2. Only the case of U^+ will be considered; the case of U^- is similar. Consider a countable sequence $([g^i_{B_i}])_{i \in \mathbb{N}} \in \mathcal{B}^+$ and let $[f_A] = \bigcirc^+_{i \in \mathbb{N}}[g^i_{B_i}]$. It needs to be shown that $\sum_{i=1}^{\infty} U^+([g^i_{B_i}]) = U^+([f_A])$. Suppose not. Since $\bigcirc^+_{i \leqslant n}[g^i_{B_i}] \precsim^+_{\sim} [f_A]$ for any n, and since U^+ is order preserving and respects finite applications of $\circ^+$, it must hold that $\sum_{i=1}^{n} U^+([g^i_{B_i}]) \leqslant U^+([f_A])$ for any n. Hence $U^+([f_A]) \geqslant \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$: given the assumption that $U^+([f_A]) \neq \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$, we have that $U^+([f_A]) > \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$.

Since there is no smallest element of $\mathcal{A}^+_{\sim}$, there exists an infinite sequence of $[h^j_{C_j}] \in \mathcal{A}^+_{\sim}$, with $[h^j_{C_j}] \succ^+_{\sim} [h^{j+1}_{C_{j+1}}]$ for all $j \in \mathbb{N}$, and a corresponding infinite decreasing sequence of real numbers $(U^+([h^j_{C_j}]))_{j \in \mathbb{N}}$. Let $[h_C]$ be the first element of the sequence with $U^+([h_C]) < U^+([f_A]) - \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$. Pick $f_A \in [f_A]$ and, using Lemma 1, an event $A' \subseteq A$ such that $f_{A'} \in [h_C]$. By construction, for all n, $\sum_{i=1}^{n} U^+([g^i_{B_i}]) \leqslant U^+([f_A]) - U^+([h_C])$, so $\bigcirc^+_{i \leqslant n}[g^i_{B_i}] \precsim^+_{\sim} [f_{A \setminus A'}]$. Using a countable number of applications of A4, generate a sequence of pairwise disjoint non-null events $A_i \subseteq A \setminus A'$ such that $f_{A_i} \in [g^i_{B_i}]$ for all i. Thus $\bigcirc^+_{i \in \mathbb{N}}[g^i_{B_i}] = [f_{A_1} \dots f_{A_i} \dots e_{(\cup A_i)^c}] \precsim^+_{\sim} [f_{A \setminus A'}] \prec^+_{\sim} [f_A] = \bigcirc^+_{i \in \mathbb{N}}[g^i_{B_i}]$ (the last equation, by the definition of $[f_A]$), which is a contradiction. The assumption that $U^+([f_A]) \neq \sum_{i=1}^{\infty} U^+([g^i_{B_i}])$ is thus false. □

Proof of Proposition 3. Suppose not: there are $[g_B]$, $[g'_{B'}] \in \mathcal{I}^-$ with $U^+(\sigma^{-1}([g'_{B'}])) \geqslant U^+(\sigma^{-1}([g_B]))$ but $[g'_{B'}] \succ_{\sim}^{-} [g_B]$. Pick $f_A \in \sigma^{-1}([g'_{B'}])$ and let $A'' \subseteq A$ be such that $f_{A''} \in \sigma^{-1}([g_B])$ (such an A'' exists by Lemma 1). Similarly, pick $g_B \in [g_B]$ with B disjoint from A, and let $B'' \subset B$ be such that $g_{B''} \in [g'_{B'}]$. Since $[g'_{B'}] \neq [g_B]$, $B \setminus B''$ is non null. By definition, $f_A g_{B''} e_{(A \cup B')^c} \sim e \sim f_{A''} g_B e_{(A'' \cup B)^c}$, and $f_A \succcurlyeq f_{A''}$. By A2, it follows that $g_{B \setminus B''} e_{(B \setminus B'')^c} \sim f_{A \setminus A''} e_{(A \setminus A'')^c} \succcurlyeq e$, contradicting the fact that $g_B \in \mathcal{A}^-$. The assumption that $U^+(\sigma^{-1}([g'_{B'}])) \geqslant U^+(\sigma^{-1}([g_B]))$ but $[g'_{B'}] \succ_{\sim}^{-} [g_B]$ is thus false. □

References

[1] Mohammed Abdellaoui, Peter Wakker, The likelihood method for decision under uncertainty, Theory Dec. 58 (2005) 3–76.

[2] Erio Castagnoli, Marco LiCalzi, Benchmarking real-valued acts, Games Econ. Behav. 57 (2006) 236–253.

[3] Itzhak Gilboa, Expected utility with purely subjective non-additive probabilities, J. Math. Econ. 16 (1987) 65–88.

[4] Brian Hill, When is there state independence? J. Econ. Theory 144 (3) (2009) 1119–1134.

[5] Edi Karni, Probabilities and beliefs, J. Risk Uncertainty 13 (1996) 249–262.

[6] Edi Karni, A theory of Bayesian decision making, Working paper, 2009.

[7] Edi Karni, David Schmeidler, Karl Vind, On state dependent preferences and subjective probabilities, Econometrica 51 (1983) 1021–1032.

[8] David H. Krantz, R. Duncan Luce, Patrick Suppes, Amos Tversky, Foundations of Measurement, vol. 1, Academic Press, San Diego, 1971.

[9] Leonard J. Savage, The Foundations of Statistics, Dover, New York, 1954, second ed. 1971.

[10] R.H. Strotz, Myopia and inconsistency in dynamic utility maximization, Rev. Econ. Stud. 23 (3) (1955) 165–180.

[11] Peter Wakker, Horst Zank, State dependent expected utility for Savage's state space, Math. Operations Res. 24 (1999) 8–34.